APPLIED MATHEMATICS AND STATISTICS LABORATORIES
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A TEST OF THE TWO-SAMPLE PROBLEM WITH NUISANCE LOCATION AND SCALE PARAMETERS, AND AN ESTIMATE OF THE SCALE PARAMETER

By

SAUL BLUMENTHAL

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I. Introduction

We consider here two related problems. The first can be viewed as an extension of the classic two-sample problem which asks whether two samples have the same underlying distribution. Thus, given two independent samples $X_1, \ldots, X_n$; $Y_1, \ldots, Y_n$ where the X_i are mutually independent with common c.d.f $F(x)$, and the Y_i are mutually independent with common c.d.f $G(x)$, the classic two-sample hypothesis is that $F(x)$ equals $G(x)$ and the alternative is that $F(x)$ and $G(x)$ are not identically equal. If the two-sample hypothesis is true and $(AX_i + B)$ is substituted for X_i $(i = 1, \ldots n)$ $(A > 0$, B real) before the test is made, any consistent test for the classic problem would reject the hypothesis with a high probability. The numbers A and B represent nuisance scale and location parameters and do not affect the form of the distribution of $(AX_i + B)$. Under some circumstances, the relevant question might be whether $F(x)$ and $G(x)$ are of the same form even though they differ through the presence of the above mentioned scale and location parameters. Formally, the hypothesis would be that $F(x) = G(Ax + B)$ for some unspecified pair A, B $(A > 0$, B real). The alternative is that the above equality does not hold. We propose a family of test statistics, all members of which lead to a consistent test of the last named hypothesis under some mild restrictions on the form of $F(x)$ and $G(x)$.

Having described the general hypothesis which we are testing, we shall suggest some specific instances where it would be useful. The range of application will be seen to be essentially the same as that of the classic two-sample test. If the experimenter wants to test whether two distributions are identical and he knows that the measurements on the two populations have been made with instruments whose zero and scale calibrations are different, then he can use the proposed test to eliminate the effect of these extraneous factors. If he knows the numerical values of the zero and scale factors and can adjust the data to a common zero and scale, then the usual tests would be more efficient.

The experimenter might consider two populations the same if they both have distributions of the same form even though they differed in a scale and location parameter. Here again the proposed test is applicable. For example, two machines might be considered interchangeable if some measurable characteristic of their output has the same distribution for each machine. It might not be feasible however at the time of making the test to adjust the machines so that the scale and location parameters of the output distributions are the same. Alternatively, through some accident, the machines might have been adjusted differently at the time the data were collected, and re-adjusted subsequently so that at the time the statistician gets the data there is no way of knowing what the relative scale and location settings were at the time of data collection. Using the proposed test, such data would not have to be discarded.

A further use of the test would be to detect a "linear drift". Suppose it is believed that observations at time zero have a distribution $F(x)$ (unspecified) and that observations at some fixed later time can be

considered to be of the form $AY + B$ where again Y has the distribution $F(x)$. Then the proposed test will reject the hypothesis if either the distribution of Y is not $F(x)$ or if the linear model $AY + B$ is not correct.

The second problem is that of estimating the scale parameter A, assuming that $F(x) = G(Ax + B)$ $(A > 0$, B real). Again a family of estimates is proposed, each member of which provides a consistent estimate of A under the same mild restrictions on $F(x)$ that are made for the test of hypothesis.

This estimation problem could arise in several ways. After accepting H_o, the experimenter might want an idea of the size of the scale parameter A. Alternatively, assuming H_o to be true on the basis of theoretical considerations, "A" might represent some physical characteristic which the experimenter wants to measure. If he is unwilling to make specific assumptions about the type of distribution involved, the proposed estimates with their fairly weak assumptions can be used. If, alternatively, the experimenter does know more about the distributions, but finds computing "best" estimates is too difficult, the proposed estimates might again be useful.

A common problem where such an estimate is needed is in the case where a time factor introduces a linear drift into the observations. If all the X's are assumed to have been measured at time t_o and the Y's at another time t_1, then the assumed model would apply.

In Section 2, we describe the proposed statistics in detail and discuss their properties. Section 3 consists of an example of the

computations involved. The following Section is concerned chiefly with establishing the consistency of the proposed estimates and tests. The important question of the distributions or even the asymptotic distributions of the proposed statistics is discussed only briefly in Section 5. Section 6 raises some additional questions.

2. The Test and The Estimator

Let $X_1, \ldots, X_n$; $Y_1, \ldots, Y_n$ be two independent sets of mutually independent random variables. Let $F(x)$ be the common c.d.f of the X_i $(i = 1, \ldots, n)$ and $G(x)$ be the common c.d.f of the Y_i $(i = 1, \ldots, n)$. We shall list in Section 4 the necessary regularity conditions on $F(x)$ and $G(x)$. Let $X_1' \leq X_2' \leq \cdots \leq X_n'$ be the ordered values of $X_1, \ldots, X_n$ and $Y_1' \leq Y_2' \leq \cdots \leq Y_n'$ be the ordered values of $Y_1, \ldots, Y_n$. Define U_i as $(X_{i+1}' - X_i')$ and V_i as $(Y_{i+1}' - Y_i')$ $(i = 1, \ldots, n-1)$. Let α, δ, p, u, v be arbitrarily chosen numbers satisfying the following restrictions:

$$0 < \alpha,\ p \leq 1\ ;\ 0 < \delta < \tfrac{1}{2}\ ;\ 0 < u < v < 1\ . \tag{2.1}$$

We then define the following statistics:

$$S_n(p,\delta) = \begin{cases} (\sin \pi p / n\pi p) \displaystyle\sum_{i=[n^{1/2+\delta}]}^{i=[n-n^{1/2+\delta}]} (U_i/V_i)^p & 0<p<1 \\[2ex] (1/n \log n) \displaystyle\sum_{i=[n^{1/2+\delta}]}^{i=[n-n^{1/2+\delta}]} (U_i/V_i) & p = 1 \end{cases} \tag{2.2}$$

$$S_n(p,u,v) = \begin{cases} (\sin \pi p / n\pi p(v-u)) \displaystyle\sum_{i=[nu+1]}^{[nv-1]} (U_i/V_i)^p & 0<p<1 \\[2ex] (1/n\ (v-u)\log n) \displaystyle\sum_{i=[nu+1]}^{[nv-1]} (U_i/V_i) & p = 1 \end{cases} \tag{2.3}$$

where $[x]$ is the greatest integer less than or equal to x.

Similarly, define $S_n'(p,\delta)$ and $S_n'(p,u,v)$ by replacing (U_i/V_i) by (V_i/U_i) in the corresponding definitions for $S_n(p,\delta)$ and $S_n(p,u,v)$.

We desire to test the hypothesis

$$(2.4) \qquad \begin{aligned} H_o &: F(x) = G(Ax + B) \\ H_1 &: F(x) \neq G(Ax + B). \end{aligned}$$

The first proposed family of tests is to reject H_o if the product $S_n(p,\delta)S_n'(p,\delta)$ is "too large". The second proposed family of tests is to reject H_o if the product $S_n(p,u,v)S_n'(p,u,v)$ is "too large". When $F(x)$ and $G(x)$ satisfy conditions (4.2) to (4.4), we show in Section 4 that $S_n(p,\delta)\ S_n'(p,\delta)$ converges stochastically to unity when H_o is true and to a quantity greater than unity otherwise. This implies that the proposed tests are consistent against alternatives $F(x)$, $G(x)$ satisfying these conditions. We show also that the tests are consistent against distributions which satisfy only condition (4.2). Derivations from condition (4.2) can lead to situations in which H_o is not true and the probability of rejecting H_o will not tend to unity as n increases. This can happen when one of the distributions has a flat section. An example is given by (4.30). If H_o is true and conditions (4.2) to (4.4) are violated, the test will tend to accept H_o in some cases and reject H_o in others.

Under the weaker assumptions (4.1) the proposed tests based on $S_n(p,u,v)\ S_n'(p,u,v)$ can be shown to be consistent by the same type of argument as above. Again, violation of this condition will lead to the situation described above. Although the $S_n(p,u,v)$ family of tests requires weaker assumptions regarding the form of $F(x)$ than does the

$S_n(p,\delta)$ family to guarantee the same operating properties, the $S_n(p,u,v)$ tests require that a fixed percentage of the observations must be ignored, whereas the percentage ignored by the $S_n(p,\delta)$ family goes to zero with n. Both families of tests are invariant under separate linear transformation on the X's and Y's. This is a reasonable requirement for any test of H_o.

The choice of p would presumably be dictated by power considerations. Power depends on distributions, and we know almost nothing about these.

Assuming that $F(x) = G(Ax + B)$, we propose to estimate the scale factor A by $S_n'(p,\delta)$ if assumptions (4.2) to (4.4) are satisfied, or by $S_n'(p,u,v)$ if condition (4.1) is satisfied. The consistency of these estimates follows from Theorem 2, Lemma 2, and Theorem 3. Again choice of p depends on the unknown distributions of the statistics.

3. Computational Example

To illustrate the type of computations involved, we shall compute the scale parameter estimate $S_n'(p, .1, .9)$, with $n = 10$ for convenience. The X's represent ten observations which were taken from a uniform distribution on the unit interval by means of a table of random numbers. The Y's represent ten observations from a uniform distribution on the interval from zero to five. Thus the factor A to be estimated is five. The table below shows the steps needed to go from the X_i and Y_i to the X_i' and Y_i' (ordered values) to the U_i and V_i (differences) to obtain finally the estimator $S_n'(p,u,v)$. Two values of p are used, $p = 1/2$ and $p = 1$ to illustrate the two types of estimators described.

i	X_i	X_i'	U_i	Y_i	Y_i'	V_i	V_i/U_i	$(V_i/U_i)^{1/2}$
1	.72586	.05275	.13538	4.44515	0.06115	0.42775	3.160	1.778
2	.30061	.18813	.04203	0.72285	0.48890	0.23395	5.566	2.359
3	.31694	.23016	.05608	2.78165	0.72285	0.62230	11.097	3.331
4	.18813	.28624	.01437	4.51455	1.34515	1.43650	99.965	9.998
5	.05275	.30061	.01633	0.06115	2.78165	0.57620	35.285	5.940
6	.79607	.31694	.03206	4.77130	3.35785	1.08730	33.914	5.824
7	.34900	.34900	.03940	0.48890	4.44515	0.06940	1.761	1.327
8	.23016	.38840	.33746	4.82835	4.51455	0.25675	0.761	0.872
9	.38840	.72586	.07021	1.34515	4.77130	0.05705	0.813	0.902
10	.28624	.79607	—	3.35785	4.82835	—	—	—

From (2.3), we see that to compute $S_{10}'(1, .1, .9)$, we must sum (V_2/U_2) through (V_8/U_8), and divide the total by $(.8)(10) \log 10$, which is 18.421. The total is 188.350 and thus our estimate $S_{10}'(1, .1, .9)$ is 10.225. To obtain $S_{10}'(1/2, .1, .9)$, we sum $(V_2/U_2)^{1/2}$ through

$(V_8/U_8)^{1/2}$ and divide by (.8)(10)(π/2) (since $\sin \pi/2 = 1$) which is 12.566. The total here is 29.651 and thus our estimate $S'_{10}(1/2, .1, .9)$ is 2.360. It is seen that one estimate is more than double the true value while the other is less than half. Thus, neither is particularly good. Since the sample size is quite small, we would not expect the asymptotic properties of the estimator to have any bearing and this example can be considered as an illustration of the danger of using a small sample when the small sample properties of the statistic are not known.

4. Convergence of the Statistics

Let $X_1,\ldots,X_n$; $Y_1,\ldots,Y_n$ and $X_1' \leq \cdots \leq X_n'$; and $Y_1' \leq \cdots \leq Y_n'$ be as described in Section 2. We shall list below regularity conditions on the distributions $F(x)$ and $G(x)$. These conditions will sometimes overlap, and not all of them need be satisfied for the stochastic convergence of a particular family of statistics. We state the conditions in terms of $F(x)$ for convenience.

(4.1) For a given pair of numbers, u_o, v_o $(0 \leq u_o < v_o \leq 1)$ $F^{-1}(u_o)$ and $F^{-1}(v_o)$ are uniquely determined, and $F(x)$ has on $[F^{-1}(u_o), F^{-1}(v_o)]$ a derivative $f(x)$ which has only a finite number of discontinuities, is bounded above, and is bounded away from zero.

(4.2) $F(x)$ has a derivative $f(x)$ which is bounded for all x, and has only a finite number of discontinuities. Further, there exists a positive value D' such that for any value D in the open interval $(0, D')$, the set of points $\{x : f(x) \geq D\}$ is an interval.

(4.3) If $f[F^{-1}(r)]$ is not bounded away from zero as r approaches unity then either there is a value $t (0 < t < 1)$ such that the quantity $d\, f[F^{-1}(r)]/dr$ is negative and non-increasing for $r > t$, or the quantity $d\, f[F^{-1}(r)]/dr$ approaches a finite negative limit as r approaches unity.

(4.4) If $f[F^{-1}(r)]$ is bounded away from zero as r approaches zero then either there is a value $t' (0 \leq t' < 1)$ such that the quantity $d\, f[F^{-1}(r)]/dr$ is positive and non-decreasing for $r < t'$, or

the quantity $d\, f[F^{-1}(r)/dr$ approaches a finite positive limit as r approaches zero.

Note that (4.2) implies (4.1). Also, the second part of (4.2) is satisfied by every unimodal $f(x)$. Conditions (4.3) and (4.4) can be verified by using the fact that $d\, f[F^{-1}(r)]/dr = d \log f[F^{-1}(r)]/dF^{-1}(r)$. Thus the condition (4.3) says that if $f(x)$ goes to zero as x increases, then there exists a finite y such that $d \log f(x)/dx$ is non-increasing for $x > y$. The conditions (4.3), (4.4) are satisfied by all infinite Polya frequency functions. The exponential, normal and Weibull distributions are well-known examples.

Now we shall consider the question of stochastic convergence. Let $h(t)$ be a bounded non-negative function of t defined for $0 \leq t \leq 1$. For p in the interval $0 \leq p \leq 1$, define $H_n(p)$ as follows:

$$(4.5) \qquad \begin{aligned} H_n(p) &= (1/n) \sum_{i=1}^{n-1} h(i/n)(U_i/V_i)^p, && 0 < p < 1 \\ H_n(p) &= (1/n \log n) \sum_{i=1}^{n-1} h(i/n)(U_i/V_i) && p = 1. \end{aligned}$$

Then we have

<u>Theorem 1.</u> If $F(x) = G(x) = x$, $(0 \leq x \leq 1)$, then as n increases, $H_n(p)$ converges stochastically to

$$(4.6) \qquad \begin{aligned} &(\pi p/n \sin \pi p) \sum_{i=1}^{n-1} h(i/n) && 0 < p < 1 \\ &(1/n) \sum_{i=1}^{n-1} h(i/n) && p = 1 . \end{aligned}$$

Proof:

Let $W_1, \ldots, W_{n-1}$ and $Z_1, \ldots, Z_{n-1}$ be two independent sets of mutually independent random variables with cdf e^{-x}. Let $R_n = \sum_{i=1}^{n-1} W_n$, $T_n = \sum_{i=1}^{n-1} Z_n$. Then it is well known that $(W_1/R_n, \ldots, W_{n-1}/R_n)$ and $(U_1, \ldots, U_{n-1})$ have the same joint density. Similarly for $(Z_1/T_n, \ldots, Z_{n-1}/R_n)$ and $(V_1, \ldots, V_{n-1})$. Also R_n is independent of $(W_1/R_n, \ldots, W_{n-1}/R_n)$ and T_n is independent of $(Z_1/T_n, \ldots, Z_{n-1}/T_n)$. Thus we need only consider the convergence of

$$(4.7) \qquad (1/n) \sum_{i=1}^{n-1} h(i/n)\, (T_n\, W_i/R_n\, Z_i)^p \qquad 0 < p < 1.$$

Using the strong law of large numbers, and making some straightforward computations, we find that $(T_n/R_n)^p$ converges to unity w.p.1 for all p, $0 < p \leq 1$. Also, for $0 < p < \frac{1}{2}$, we find that

$$(4.8) \qquad (1/n) \sum_{i=1}^{n-1} h(i/n)\, (W_i/Z_i)^p$$

converges to (4.6) w.p.1. For $\frac{1}{2} \leq p < 1$, the results in Gnedenko and Kolmogorof [2] on the relative stability of a sum of positive random variables with finite expectations show that (4.8) converges stochastically to (4.6).

For the case $p = 1$, we must consider

$$(4.9) \qquad (1/n \log n) \sum_{i=1}^{n-1} h(i/n)\, (W_i/Z_i) .$$

Again the convergence of (4.9) stochastically to (4.6) follows from results in [2] on asymptotic stability of sums of positive random variables. The boundedness of $h(t)$ is important in all of the above convergence computations. This proves Theorem 1.

Corollary 1. If $F(x) = G(x) = x$ $(0 \leq x \leq 1)$, and $\int_0^1 h(t)dt$ exists (in the Riemann sense), as n increases, $H_n(p)$ converges stochastically to

$$(4.10)\qquad \begin{cases} (\pi p/\sin \pi p) \int_0^1 h(t)dt & 0 < p < 1 \\ \int_0^1 h(t)dt & p = 1 . \end{cases}$$

Proof:

If $\int_0^1 h(t)dt$ exists, then as n increases, (4.6) approaches (4.10). This proves the corollary.

Now suppose condition (4.1) is satisfied and let $S_n(p,u,v)$ be defined by (2.3) with $0 \leq u_o < u < v < v_o \leq 1$, where u_o, v_o are given in (4.1) Then we have

Theorem 2. As n increases, $S_n(p,u,v)$ converges stochastically to

$$(4.11)\qquad (1/(v-u))\int_u^v (g[G^{-1}(t)]/f[F^{-1}(t)])^p \, dt$$

Proof:

If X'_{j+1} and X'_j are both in $[F^{-1}(u),\ F^{-1}(v)]$, and Y'_{j+1} and Y'_j are both in $[G^{-1}(u),\ G^{-1}(v)]$, we can write

$$(4.12)\qquad \begin{aligned} F(X'_{j+1}) - F(X'_j) &= f(\theta_j)(X'_{j+1} - X'_j) \ , \ (X'_j \leq \theta_j \leq X'_{j+1}) \\ G(Y'_{j+1}) - G(Y'_j) &= g(\theta'_j)(Y'_{j+1} - Y'_j) \ , \ (Y'_j \leq \theta'_j \leq Y'_{j+1}) \ . \end{aligned}$$

Define λ_j as $|F(X'_j) - (j/n)|$ and λ'_j as $|G(Y'_j) - (j/n)|$. By the Glivenko-Cantelli theorem, we know that for any positive δ, $\max_i n^{1/2-\delta}\lambda_i$ converges stochastically to zero as n increases. Define δ_j as $|X'_j - F^{-1}(j/n)|$. If X'_j and $F^{-1}(j/n)$ are both in the interval $[F^{-1}(u), F^{-1}(v)]$, then since $f(x) \geq A > 0$ on this interval (by (4.1)), we have $\delta_j \leq (\lambda_j/A)$. Then if X'_j, X'_{j+1}, $F^{-1}(j/n)$ and $F^{-1}((j+1/n)$ are all in the interval $[F^{-1}(u), F^{-1}(v)]$, we have that $|\theta_j - F^{-1}(j/n)| \leq \delta_j + \delta_{j+1} + 1/nA$, and we can write

$$F(X'_{j+1}) - F(X'_j) = f[F^{-1}(j/n)]\,(X'_{j+1} - X'_j) + \gamma_j(X'_{j+1} - X'_j) \tag{4.13}$$

where $\gamma_j = f(\theta_j) - f[F^{-1}(j/n)]$. But because of the uniform continuity of $f(x)$ in $[F^{-1}(u), F^{-1}(v)]$, the above bound for $|\theta_j - F^{-1}(j/n)|$ and the Glivenko-Cantelli theorem, it is easily seen that $\max_{nu<i<nv} |\gamma_i|$ converges stochastically to zero as n increases. Similarly we can show that

$$G(Y'_{j+1}) - G(Y'_j) = g[G^{-1}(j/n)](Y'_{j+1} - Y'_j) + \gamma'_j(Y'_{j+1} - Y'_j) \tag{4.14}$$

where $\max_{nu<i<nv} |\gamma'_i|$ converges stochastically to zero as n increases. Denote $F(X'_{j+1}) - F(X'_j)$ by $W_j (j = 1,\ldots,n-1)$, and $G(Y'_{j+1}) - G(Y'_j)$ $(j = 1,\ldots,n-1)$ by Z_j. Notice that $(W_1,\ldots,W_{n-1})$ and $(Z_1,\ldots,Z_{n-1})$ are distributed as two independent sets of sample successive differences from the uniform distribution on the unit interval. Using (4.13) and (4.14) we have

$$\begin{aligned} \sum_{nu<i<nv} (g[G^{-1}(i/n)]/f[F^{-1}(i/n)])^p\,(W_i/Z_i)^p &= \sum_{nu<i<nv} (U_i/V_i)^p + \\ &+ \sum_{nu<i<nv} (U_i/V_i)^p\,\{(f(\theta_i)\,g[G^{-1}(i/n)]/g(\theta'_j)f[F^{-1}(i/n)])^p - 1\}\,. \end{aligned} \tag{4.15}$$

Using the result of Corollary 1, we have that the left side of (4.15) when properly normalized converges stochastically to (4.11). Let the appropriately normalized second term on the right side of (4.15) be represented by $\bar{\bar{S}}_n(p,u,v)$. As n increases, the probability approaches unity that $|\bar{\bar{S}}_n(p,u,v)|$ will be no greater than

$$S_n(p,u,v)\ \{(\max_{nu<i<nv} |\gamma_i|^p D^p + \max_{nu<i<nv} |\gamma_i'|^p|B^p)/A^pC^p\}$$

where $0 < A \leq f(x) \leq B < \infty$ for $F^{-1}(u) \leq x \leq F^{-1}(v)$, and $0 < C \leq g(x) \leq D < \infty$, for $G^{-1}(u) \leq x \leq G^{-1}(v)$. Thus we have that $|\bar{\bar{S}}_n(p,u,v)|/S_n(p,u,v)$ converges stochastically to zero as n increases. But $S_n(p,u,v) + \bar{\bar{S}}_n(p,u,v)$ converges stochastically to (4.11) as n increases. This proves Theorem 2.

It is clear that the same proof will show that under condition (4.1), $S_n'(p,u,v)$ (defined in Section 2) converges stochastically to

$$(4.15) \qquad (1/(v-u)) \int_u^v (f[F^{-1}(t)]/g[G^{-1}(t)])^p\, dt.$$

Thus, under condition (4.1), we obtain that as n increases the statistic $S_n'(p,u,v)\ S_n(p,u,v)$ converges stochastically to

$$(4.16) \qquad (1/n(v-u))^2 \left(\int_u^v (f[F^{-1}(t)]/g[G^{-1}(t)])^p dt\right)\left(\int_u^v (g[G^{-1}(t)]/f[F^{-1}(t)])^p dt\right).$$

Now, assume that conditions (4.2) to (4.4) are satisfied. Let $S_n(p,\delta)$ be given by (2.2). Then we have

<u>Theorem 3.</u> If $F(x) = G(Ax + B)$, then as n increases, $S_n(p,\delta)$ converges stochastically to $(1/A)^p$.

Proof:

We assume that either or both of the quantities $\lim_{x \downarrow F^{-1}(0)} f(x)$, $\lim_{x \uparrow F^{-1}(1)} f(x)$ is zero. Otherwise, Theorem 2 applies. For arbitrarily chosen u, v, $0 < u < v < 1$, we have

$$S_n(p,\delta) = (v-u)\, S_n(p,u,v) + b(n,p) \left(\sum_{n^{1/2+\delta} < i \le nu} (U_i/V_i)^p + \sum_{nv \le i < n-n^{1/2+\delta}} (U_i/V_i)^p \right) \tag{4.19}$$

where

$$b(n,p) = \begin{cases} \sin \pi p / n\pi p & 0 < p < 1 \\ 1/n \log n & p = 1 \,. \end{cases} \tag{4.20}$$

By Theorem 2, $(v-u)\, S_n(p,u,v)$ converges stochastically to

$$\int_u^v (g[G^{-1}(t)]/f[F^{-1}(t)])^p \, dt = (v-u)(1/A)^p \,. \tag{4.21}$$

We shall now investigate $\sum_{nv \le i < n-n^{1/2+\delta}} (U_i/V_i)^p$, the treatment of $\sum_{n^{1/2+\delta} < i \le nv} (U_i/V_i)^p$ being entirely similar.

Since $F(x) = G(Ax + B)$, we see that $(Y_1,\ldots,Y_{n-1})$ has the same distribution as $(A\bar{Y}_1 + B,\ldots,A\bar{Y}_n + B)$ where $\bar{Y}$ has the distribution $F(x)$. Letting the ordered $\bar{Y}_i$ be $\bar{Y}_1' \le \ldots \le \bar{Y}_n'$, and the differences be $V_i' = \bar{Y}_{i+1}' - Y_i'$ $(i=1,\ldots,n-1)$, we have that $\sum (U_i/V_i)^p$ has the same distribution as $\sum (U_i/AV_i')^p$. Thus we want to study the convergence of $\sum_{nv \le i < n-n^{1/2+\delta}} (U_i/V_i')^p$. Using condition (4.3) and the relation (4.12), we obtain that $\sum_{nv \le i < n-n^{1/2+\delta}} (U_i/V_i)^p$ can be bounded above by

(4.22) $$\sum_{nv \le i < n-n^{1/2+\delta}} (f(\bar{Y}_i')/f(X_{i+1}'))^p \{(F(X_{i+1}')(-F(X_i'))/(F(\bar{Y}_{i+1}')-F(\bar{Y}_i'))\}^p$$

If we can find a bound (say B) for $f(\bar{Y}_i')/f(X_{i+1}')$, then we would be through, for (4.22) would then be bounded by

(4.23) $$B^p \sum_{nv \le i \le n} \{(F(X_{i+1}'))/(F(\bar{Y}_i') - (F(\bar{Y}_i'))\}^p \quad .$$

By Corollary 1, (4.23) when normalized to agree with $S_n(p,\delta)$ converges stochastically to $(1-v)B^p$. Thus, by choosing v^* (say) sufficiently large, we will have that

$$P\ \{b(n,p) \sum_{nv \le i < n-n^{1/2+\delta}} (U_i/V_i)^p \le \epsilon\}$$

approaches unity as n increases for each v such that $v^* < v \le 1$.

To bound $f(Y_i')/f(X_{i+1}')$, we note that if $\bar{Y}_i' \ge X_{i+1}'$, then unity serves as a bound. Thus we must find a bound (B') for this ratio when $\bar{Y}_i' < X_{i+1}'$. We can then take $B = 1 + B'$. Letting $t_1 = F(\bar{Y}_i')$ and $t_2 = F(X_{i+1}')$ (momentarily suppressing i), we have $t_1 < t_2$, and

(4.24) $$(f(\bar{Y}_i')/f(X_{i+1}')) = (f[F^{-1}(t_1)]/f[F^{-1}(t_2)]).$$

By the mean value theorem, and the fact that $f[F^{-1}(1)] = 0$, we have

(4.25)
$$f[F^{-1}(t_1)] = (1-t_1)(-df[F^{-1}(t)]/dt)_{t=\theta_1} \qquad t_1 \le \theta_1 \le 1$$
$$f[F^{-1}(t_2)] = (1-t_2)(-df[F^{-1}(t)]/dt)_{t=\theta_2} \qquad t_2 \le \theta_2 \le 1 \ .$$

From condition (4.3), we see that there is a value v_1 such that if t_1, t_2 both exceed v_1, then either $(df[F^{-1}(t)]/dt)_{t=\theta_j}$ $(j=1,2)$ is

sufficiently near a limit L, in which case the ratio (4.24) is bounded by $2(1-t_1)/(1-t_2)$, or $df[F^{-1}(t)]/dt$ is a decreasing function of t for $t \geq t_1$. In the latter case, by Lemma 1 below, we have $(1-t_1)/(1-t_2)$ as a bound for the ratio (4.24). Clearly, if $v > v_1$, then as n increases,

$$P\ \{F(\bar{Y}_i') > v_1,\ F(X_{i+1}') > v_1\ ,\ nv \leq i \leq n\}$$

approaches unity. Thus, we need only to find a bound for $[1-F(\bar{Y}_i')]/[1-F(X_{i+1}')]$, $(nv \leq i < n-n^{1/2+\delta})$. Rewriting this last ratio as

$$\frac{\left(\frac{n-i}{n}\right) n^{1/2-\delta} + n^{1/2-\delta}\left(\frac{i}{n} - F(\bar{Y}_i')\right)}{\left(\frac{n-i-1}{n}\right) n^{1/-\delta} + n^{1/2-\delta}\left(\left(\frac{i+1}{n}\right) - F(X_{i+1}')\right)}$$

using the Glivenko-Cantelli Theorem (see Theorem 2), and noting that $[(n-i)/n^{1/2+\delta}]$ is greater than unity, we have that

$$P\ \{([1-F(\bar{Y}_i)]/[1-F(X_{i+1}')] < 2\ ,\ nv \leq i < n-n^{1/2+\delta}\}$$

approaches unity as n increases.

Putting together all of the above pieces, and noting that we can choose u,v so that $(v-u)$ is arbitrarily close to unity, we see that

$$P\ \{|S_n(p,\delta) - (1/A)^p\ | < \epsilon\}$$

approaches unity as n increases. This proves Theorem 3.

Lemma 1. Suppose $f(x)$ is continuous and differentiable on the interval (a,b), with $f'(x) < 0$ and decreasing. Let $x_1 < x_2$ be in (a,b). By the mean value theorem,

$$(4.26) \qquad f(x_i) = f(b) + (x_i - b)\, f'(\theta_i) \qquad x_i \leq \theta_i \leq b\ ,\ i = 1,2\ .$$

Then $\qquad \theta_2 \geq \theta_1$.

<u>Proof:</u>

First note that

$$(4.27) \qquad f(x_1) = f(x_2) - (x_2 - x_1)\,(f'(\theta_3)) \qquad x_1 \leq \theta_3 \leq x_2$$

and that $\qquad \theta_3 \leq \theta_1$. If $\theta_1 < \theta_3$, then

$$(4.28) \qquad \begin{aligned} f(x_1) &= f(b) + (b-x_1)(-f'(\theta_1)) < f(b) + (b-x_1)(-f(\theta_3)) = \\ &= f(b) + (b-x_2)(-f'(\theta_3)) + (x_2 - x_1)(-f'(\theta_3))\ . \end{aligned}$$

Using (4.27) in (4.28), and noting (4.26) we have $\theta_3 > \theta_2$ which is impossible. Writing $f(x_1) = f(x_2) + (f(x_1) - f(x_2))$, then using (4.26) to represent $f(x_2)$ and (4.27) to represent $(f(x_1) - f(x_2))$ and subtracting the result from the representation of $f(x_1)$ given by (4.26), we obtain

$$(4.29) \qquad \begin{aligned} 0 &= (x_1 - b)\, f'(\theta_1) - (x_2 - b)\, f'(\theta_2) + (x_2 - x_1) f'(\theta_3) \\ &= (x_2 - b)(f'(\theta_1) - f'(\theta_2)) + (x_2 - x_1)(f'(\theta_3) - f'(\theta_1)). \end{aligned}$$

Since $\theta_3 \leq \theta_1$, $f'(\theta_1) - f'(\theta_2)$ is positive, or $\theta_1 \leq \theta_2$. This proves Lemma 1.

In the same way that we proved Theorem 2, we can show that under the same conditions, $S_n'(p,\delta)$ converges stochastically to A^p. Putting together these results, we have that $S_n(p,\delta)\ S_n'(p,\delta)$ converges stochastically to unity as n increases, when $F(x) = G(Ax + B)$, and conditions (4.2) to (4.4) are valid.

We shall now state a Lemma due to Weiss [3], which will be useful below.

Lemma 2. If $F(x)$ and $G(x)$ are two distribution functions and u, v $(0 \leq u < v \leq 1)$ are two given numbers, suppose $F^{-1}(u)$, $F^{-1}(v)$, $G^{-1}(u)$, $G^{-1}(v)$ are all uniquely determined. Also suppose that $F(x)$ has a derivative $f(x)$ between $F^{-1}(u)$ and $F^{-1}(v)$, and $G(x)$ has a derivative $g(x)$ between $G^{-1}(u)$ and $G^{-1}(v)$. Then a sufficient condition that $f[F^{-1}(r)] = kg[G^{-1}(r)]$ for almost all r in $[u,v]$ (where k is a positive constant) is that there are two constants C, D $(C > 0)$, such that $F(Cx + D) = G(x)$ for all x in the interval $[G^{-1}(u), G^{-1}(v)]$. If in addition, $f(x) > 0$ between $F^{-1}(u)$ and $F^{-1}(v)$, the condition is necessary as well as sufficient.

We omit the proof since it is contained in [3]. Note that Weiss omitted the condition $f(x) > 0$ between $F^{-1}(u)$ and $F^{-1}(v)$ and without this condition the statement is incorrect. A simple counter-example is given by the following pair of distributions:

$$
\left\{
\begin{array}{lll}
G(x) & = 0 & x \leq 0 \\
 & = 2x(1-x) & 0 \leq x \leq 1/2 \\
 & = 1-2x(1-x) & 1/2 \leq x \leq 1 \\
 & = 1 & 1 \leq x \\
F(x) & = 0 & x \leq 0 \\
 & = 4x(1-2x) & 0 \leq x \leq 1/4 \\
 & = 1/2 & 1/4 \leq x \leq 3/4 \\
 & = 1-4(1-x)(2x-1) & 3/4 \leq x \leq 1 \\
 & = 1 & 1 \leq x
\end{array}
\right. \tag{4.30}
$$

Here $(f[F^{-1}(r)]/g[G^{-1}(r)]) = 2$ for all r in $[0,1]$, except $r = 1/2$, but clearly $G(x) \neq F(Cx + D)$ for any pair C, D $(C > 0)$.

Now we shall return to the problem of examining $S_n'(p,\delta)\ S_n(p,\delta)$ when conditions (4.2) to (4.4) hold but $F(x) \neq G(Ax + B)$. We have

Theorem 4. If $F(x) \neq G(Ax + B)$ $(A > 0)$, and conditions (4.2) to (4.4) obtain, then as n increases, for every $\epsilon > 0$,

$$(4.31) \qquad P\ \{S_n'(p,\delta)\ S_n(p,\delta) + \epsilon \geq [\int_0^1 (f[F^{-1}(t)]/g[G^{-1}(t)])^p\ dt] \cdot$$

$$\cdot [\int_0^1 (g[G^{-1}(t)]/f[F^{-1}(t)])^p\ dt]\}$$

approaches unity.

Proof:

Let $S_n(p,\delta)$ and $S_n'(p,\delta)$ be represented as in (4.19). Then note that $S_n'(p,\delta)\ S_n(p,\delta)$ can be written as

$$(4.32) \qquad (v-u)^2\ S_n'(p,u,v)\ S_n(p,u,v) + B(n,p,u,v)$$

where u,v $(0 < u < v < 1)$ are arbitrary and $B(n,p,u,v)$ is strictly positive. By (4.17) we have that $(v-u)^2\ S_n'(p,u,v)\ S_n(p,u,v)$ converges stochastically to

$$(4.33) \qquad \int_u^v h(t)\ dt \int_u^v (1/h(t))\ dt$$

where

$$(4.34) \qquad h(t) = (f[F^{-1}(t)]/g[G^{-1}(t)])^p\ .$$

But by the Lebesgue Monotone Convergence theorem, we know that by making $(v-u)$ close to unity, we can, for any positive ϵ, make (4.33) greater than

$$(4.35) \qquad \int_0^1 h(t)\, dt \int_0^1 (1/h(t))dt - \epsilon \ .$$

This completes the proof of Theorem 4.

Assuming the conditions (4.2) to (4.4) we can now prove the consistency of the test which rejects H_o when $S_n'(p,\delta)\ S_n(p,\delta)$ is "too large." By Lemma 2, we have that when H_o is not true, $h(t)$ (given by 4.34) is not a constant on $[0,1]$. It can then be shown that

$$(4.36) \qquad \int_0^1 h(t)dt \int_0^1 (1/h(t))dt$$

is greater than unity. Thus using Theorem 4, we have that

$$(4.37) \qquad P\ \{S_n'(p,\delta)\ S_n(p,\delta) > 1\}$$

approaches unity as n increases. This establishes consistency when conditions (4.2) to (4.4) are satisfied. It is easily seen that conditions (4.3) and (4.4) were not essential to the argument. Thus even if only condition (4.2) holds, the test will be consistent. If condition (4.2) is violated, we must allow distributions such as those given in (4.30) and for such distributions it can not be shown that the probability (4.37) approaches unity as n increases.

5. Remarks About Large Sample Distributions.

We shall give a heuristic argument to indicate why for some values of p, $S_n'(p,u,v)$ and $S_n(p,u,v)$ might have a bivariate normal limiting distribution, whereas for other values of p this distribution is not possible. Similar statements can be made about $S_n'(p,\delta)$ and $S_n(p,\delta)$. From the proof of Theorem 2, we see that $S_n'(p,u,v)$ and $S_n(p,u,v)$ have approximately the same joint distribution as

$$b(n,p) \sum_{i=nv}^{nv} \{f[F^{-1}(i/n)]/g[G^{-1}(i/n)]\}^p \quad (W_i/Z_i)^p \quad \text{and}$$

(5.1)

$$b(n,p) \sum_{i=nu}^{nv} \{g[G^{-1}(i/n)]/f[F^{-1}(i/n)]\}^p \quad (Z_i/W_i)^p$$

where $(W_1,\ldots,W_{n-1})$ and $(Z_1,\ldots,Z_{n-1})$ have the same joint distributions as the corresponding quantities in Theorem 2, and $b(n,p)$ is given by (4.20).

From the proof of Theorem 1, we see that the quantities in (5.1) have approximately the same joint distribution as

$$b(n,p) \sum_{i=nu}^{nv} (h(i/n))^p \, (\bar{X}_i/\bar{Y}_i)^p \qquad \text{and}$$

(5.2)

$$b(n,p) \sum_{i=nu}^{nv} (1/h(i/n))^p \, (\bar{Y}_i/\bar{X}_i)^p$$

where

(5.3) $$h(i/n) = f[F^{-1}(i/n)]/g[G^{-1}(i/n)]$$

and $(\bar{X}_1, \ldots, \bar{X}_n)$ are independent random variables with c.d.f e^{-x} and $(\bar{Y}_1, \ldots, \bar{Y}_n)$ are independent random variables with c.d.f e^{-y}.

Each of the quantities in (5.2) is a linear combination of independent random variables, and when $p < 1/2$, the first two moments of these variables are finite. Thus, the bivariate central limit theorem applies and the expressions in (5.2) have a limiting bivariate normal distribution. Thus, for $p < 1/2$, it is reasonable to suppose that $S_n'(p,u,v)$ and $S_n(p,u,v)$ have a limiting bivariate normal distribution. When $p = 1/2$, all moments of $(\bar{X}_i/\bar{Y}_i)$ up to the second exist, and each term in (5.2) can be shown to have a limiting normal distribution. Similarly, a bivariate limiting normal distribution for the terms in (5.2) can be obtained from a generalization of the one-dimensional result.

When $p > 1/2$, not all moments of $(\bar{X}_i/\bar{Y}_i)^p$ of order less then two exist. If one of the expressions in (5.2) had a limiting normal distribution, then the expression

$$(5.4) \qquad b(n,p) \sum_{i=nu}^{nv} (\bar{X}_i/\bar{Y}_i)^p$$

where $h(i/n)$ is taken to be identically unity would have a normal limiting distribution. However, (5.4) is a sum of independent, identically distributed random variables, and from Cramér[1], Theorem 23, we find that a necessary condition that such a sum have a limiting normal distribution is the existance of all moments of order less than two. Thus, for $p > 1/2$, it is reasonable to say that in general $S_n'(p,u,v)$ and $S_n(p,u,v)$ will not have a limiting bivariate normal distribution. What sort of limiting distribution these quantities do

have when $p > 1/2$, is completely open.

Thus from the viewpoint of being able to say something about asymptotic power, there is some advantage to using the tests with $p \leq 1/2$ rather than those with $p > 1/2$.

6. Further Problems.

Since each member of the family $S_n(p,u,v)$ (for u,v fixed) is a consistent estimator of $(1/A)$, it follows that any linear combination of a finite number of these estimators (with the weighting factors totaling unity) will also be a consistent estimator of $(1/A)$. Similarly, one would guess that if $H(p)$ is a probability distribution on $[0,1]$ then

$$\int_0^1 (S_n'(p,u,v)dH(p)$$

should also be a consistent estimator of $(1/A)$. It might be worth considering whether there is some $H(p)$ which in some sense gives a "better" estimator than any individual $S_n(p,u,v)$. Similar remarks apply to $S_n(p,\delta)$. In this case, the limiting behavior of $S_n(p,\delta)$ as $\delta \to 0$ would also be of interest.

It is possible to modify the two-sample problem treated herein so that it becomes a two population test of fit. Namely, suppose that $H(x)$ is a given distribution function. Then under H_0 we have $F(x) = G(Ax + B) = H(Cx + D)$ where A, B, C, D are real but unspecified constants $(A > 0,\ C > 0)$. The question here is not whether $F(x)$ and $G(x)$ are the same "type" of distribution but rather whether they are both the same specific type. This modified problem can be solved by using the statistic Z_n proposed by Weiss [3]. Let $Z_n(X)$ be defined for fixed u, v $(0 \leq u < v \leq 1)$ by

$$(6.1) \qquad Z_n(X) = \frac{n \sum_{nu < j < nv} h^2 [H^{-1}(j/n)](X_{j+1} - X_j)^2}{\{ \sum_{nu<j<nv} h[H^{-1}(j/n)](X_{j+1} - X_j)\}^2}$$

and let $Z_n(Y)$ be defined similarly. Then from the results of [3], it is easily seen that under condition (4.1), both of the following tests are consistent

1) Reject H_o if $Z_n(X) Z_n(Y)$ is "too large", or

2) Reject H_o if $Z_n(X) + Z_n(Y)$ is "too large".

Which of these two tests is better depends on their limiting power, which in turn depends on the unknown limiting distributions of the proposed statistics.

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References

[1] H. Cramér, Random Variables and Probability Distributions, Cambridge University Press, 1937.

[2] B. V. Gnedenko and A. N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, Addison-Wesley, 1954.

[3] L. Weiss, "Tests of Fit in the Presence of Nuisance Location and Scale Parameters," Ann. Math. Stat., vol. 28 (1957), pp. 1016-1020.

[4] L. Weiss, "On the Estimation of Scale Parameters," Naval Res. Logistics Quart., vol. 8 (1961), pp. 245-256.

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